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A Generalization of the Cagniard Method

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The Cagniard method for obtaining the inverse Laplace transform of integrals, used when solving wave-propagation problems by generalized rays, was meant originally for simple cases of point-sources with a step-function time-dependence and simple structures. Gradually, the method was extended to more complex sources and structures but in many cases the solution involved expressions requiring convolutions. The extension presented here enables one to obtain the time-dependent solution for various complex cases in a form similar to that for simple cases, i.e., in terms of simple integrals, without convolutions. Several examples are given: a strike-slip point-source with linear time-dependence, a dip-slip point-source with linear time-dependence, and a strike-slip point-source with quadratic time-dependence.

1. INTRODUCTION

In problems of elastic wave-propagation from point-sources and also from sources of finite extent, an essential role is played by the inversion of Laplace transforms representing generalized rays, of the form

$$\mathscr{F}_m^{n,s}(\varphi, p) = k^n \int_0^\infty x^{l+s}\varphi(x)\, J_m^{(s)}(krx)\, e^{-khg(x)}\, dx. \tag{1.1}$$

Here

$$\begin{aligned} l &= 1 \quad \text{for} \quad m = \text{even}, \\ &= 0 \quad \text{for} \quad m = \text{odd}, \end{aligned} \tag{1.2}$$

$$g(x) = \sum_{j=1}^N \omega_j(x^2 + \lambda_j^2)^{1/2}, \qquad 0 \leqslant \omega_j \leqslant 1, \quad 0 < \lambda_j \leqslant 1, \tag{1.3}$$

h and r are constants, the first usually representing the overall depth of the layer in a layered structure while the second represents the horizontal distance between source and receiver. $J_m^{(s)}$ is the s-order derivative of the mth order Bessel function, n is an integer, and

$$k = p/c, \tag{1.4}$$

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p being the transform variable and c a characteristic wave velocity. The function $\varphi(x)$ is a rational function of x^2 and the square roots $(x^2 + \lambda_j^2)^{1/2}$ and possibly some additional square roots of the same form $(x^2 + \bar{\lambda}_k^2)^{1/2}$, $k = 1, K$.

The transform is taken here as

$$\bar{f}(p) = p \int_0^\infty e^{-pt} f(t)\, dt. \tag{1.5}$$

Studying the propagation of elastic waves in a medium consisting of two half-spaces, Cagniard [7] was the first to show how to use techniques from the theory of analytic functions in order to bring (1.1) in the particular case $m = 0$, $n = 1$, $N = 2$, $s = 0$, to a form on which it is easy to recognize the transform of a known function. For the particular cases $m = 0$, $n = 1$, $N = 1$, $s = 0,1$, similar techniques were used by Pekeris for an SH-torque pulse in [18] and for a vertical force in a half-space in [19, 20, 21]. A different approach for obtaining the inverse of (1.1) for an impulsive line-force in an elastic half-space was presented by Sherwood [23] whereas a modification of Cagniard's method was given by de Hoop [10, 11] and used by Helmberger [9] for a more general case ($N > 2$) corresponding to rays undergoing multiple reflections or refractions. Cagniard's original method was combined by Longman [14] with the Pekeris version and used for the propagation of an SH-pulse in a layered solid by Pekeris, Alterman, and Abramovici [22] and for a P-pulse in a layered solid by Pekeris, Alterman, Abramovici, and Jarosh [23], Abramovici and Alterman [2], and by Abramovici [1]. For a vertical force in a layered solid this method was used by Abramovici and Gal-Ezer [3]. In these papers the inversion of (1.1) was obtained for $N = 2$, $n = 1$ but also for $n = 0$ corresponding to a source having a linear time-dependence, whereas in all the previous work only the case $n = 1$ was treated, i.e., that corresponding to a source with a Heaviside step-function time-dependence. The inversion of integrals of the form (1.1) for $n = 1$ for explosive sources, vertical forces, and double couples were obtained also by G. Müller in [15, 16, 17] for multilayered structures. For an impulsive double couple the inversion of (1.1) for a general N was given also by Chandra in [8]. Using Helmberger's approach, Ben-Menahem and Vered [6] extended Cagniard's method to general multipolar sources obtaining therefore the inversion of (1.1) for a general N and m.

In the present paper we make use of Cagniard's original approach in Longman's version to show how to obtain the inverse transform of the integral (1.1) for any integer n. We present first the case $n = 1$ for a multipolar source in a layered medium and then show how to use the obtained formulas for the case $n = 0$ and recursively for any negative n. The cases $n \leqslant 0$ are needed for linear or quadratic sources, for avoiding the use of convolutions in calculating some of the displacement components for certain types of point-sources [25] and for calculating the displacement for finite sources [12, 13, 4].

2. The Inverse Transform for $n = 1$

Following Cagniard's procedure closely, use an integral representation for $J_m(z)$ [5]

$$J_m(z) = i^{-m}/\pi \int_0^{\pi} e^{iz\cos\theta} \cos m\theta \, d\theta \tag{2.1}$$

leading to

$$\begin{aligned} J_m(z) &= (-1)^{m/2} \frac{2}{\pi} \mathcal{R}e \int_0^{\pi/2} e^{-iz\cos\theta} \cos m\theta \, d\theta \qquad \text{for} \quad m = \text{even} \\ &= (-1)^{(m+1)/2} \frac{2}{\pi} \mathcal{I}m \int_0^{\pi/2} e^{-iz\cos\theta} \cos m\theta \, d\theta \qquad \text{for} \quad m = \text{odd}. \end{aligned} \tag{2.2}$$

Differentiating s times we get

$$\begin{aligned} J_m^{(s)}(z) &= (-1)^{m/2} \frac{2}{\pi} \mathcal{R}e \int_0^{\pi/2} (-i\cos\theta)^s e^{-iz\cos\theta} \cos m\theta \, d\theta \qquad \text{for} \quad m = \text{even} \\ &= (-1)^{(m+1)/2} \frac{2}{\pi} \mathcal{I}m \int_0^{\pi/2} (-i\cos\theta)^s e^{-iz\cos\theta} \cos m\theta \, d\theta \qquad \text{for} \quad m = \text{odd}. \end{aligned} \tag{2.3}$$

Thus, leaving aside a multiplicative constant and postponing taking the real or the imaginary part, we want to find the inverse transform of

$$T(p) = k \int_0^{\pi/2} (-i\cos\theta)^s \cos m\theta \, d\theta \int_0^{\infty} f(x)\, e^{-kh[g(x)+i\rho x\cos\theta]} \, dx, \tag{2.4}$$

where

$$\rho = r/h \tag{2.5}$$

and

$$f(x) = x^{l+s}\varphi(x) \tag{2.6}$$

Stage 1. *Change of Variable*

Following Cagniard, the idea is to find a change of variable that will bring (2.4) to a form on which the transform of a known function is easily recognizable. A natural candidate would be the exponent in (2.4)

$$y = g(x) + i\rho x \cos\theta. \tag{2.7}$$

The following result shows that $y = y(x)$ may indeed be taken as a change of variable.

Theorem 2.1. *The application $y = y(x)$ is one-to-one for* $\mathrm{Re}(x) > 0$ *and* $\mathrm{Re}(y) > g(0)$ *if:*

(a) *the x-plane is cut along the imaginary axis between* $-\lambda_j i$ *and* $\lambda_j i$,

(b) *the square roots* $(x^2 + \lambda_j^2)^{1/2}$ *are defined as having a positive real part for* $\mathrm{Re}(x) > 0$,

(c) *the y-plane is cut along the real axis between E and M (Fig. 1) corresponding to* $y(i\lambda)$ *and* $y(iv_0)$, *where*

$$\lambda = \min (\lambda_j) \tag{2.8}$$

and $v_0 > 0$ *is the smallest real zero of the derivative of the function*

$$\psi(v) = y(iv) = \sum_{j=1}^{N} \omega_j(\lambda_j^2 - v^2)^{1/2} - \rho v \cos \theta. \tag{2.9}$$

Proof. Use the argument principle for a contour in the x-plane formed by a semicircle in the first quadrant having the origin as center and of sufficiently large radius, and the corresponding portion of the imaginary axis (Fig. 2).

Performing the change of variable in the internal integral (2.5) we get

$$T(p) = k \int_0^{\pi/2} (-i \cos \theta)^s \cos m\theta \, d\theta \int_\Gamma f[x(y, \theta)] \, e^{-khy} \frac{\partial x(y, \theta)}{\partial y} \, dy, \tag{2.10}$$

where $x = x(y, \theta)$ is the inverse of (2.7) and the contour Γ (Fig. 2) is a path in the

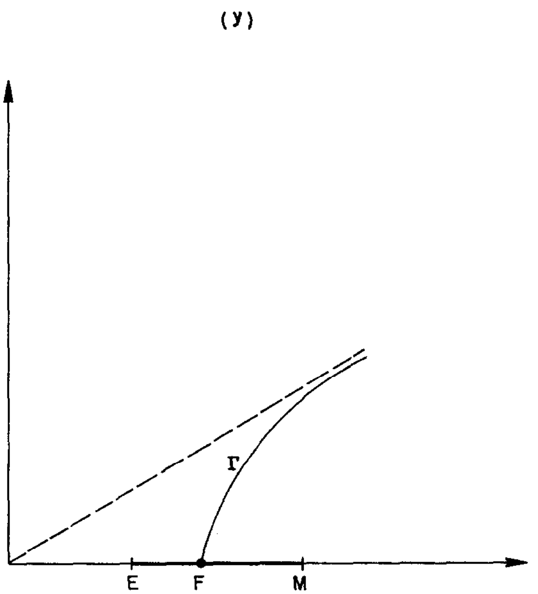

FIG. 1. The complex plane for the variable $y = g(x) + i\rho x \cos \theta$.

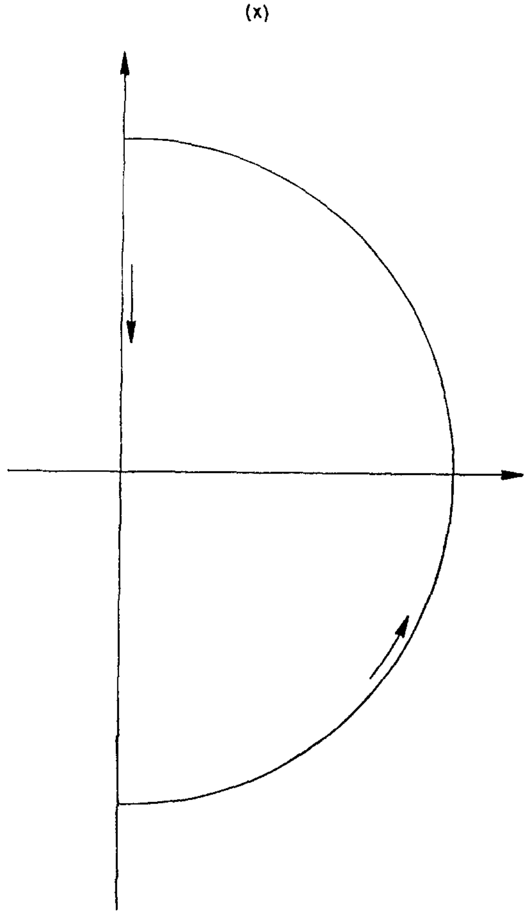

FIG. 2. The complex plane for the original variable x.

first quadrant starting at F corresponding to $\bar{y} = y(0)$ and going to infinity asymptotically approaching the line

$$y = \left(\sum_{j=1}^{N} \lambda_j + ip \cos \theta\right) \mathscr{R}e(x). \tag{2.11}$$

Stage 2. *Change of Integration Path*

If $f(x)$ does not have singular points in the domain included between Γ and the real axis, which is the case when (1.1) represents generalized rays in layered media, one can use Caucy's theorem and replace Γ by the real axis from F to infinity

$$T(p) = k \int_0^{\pi/2} (-i \cos \theta)^s \cos m\theta \, d\theta \int_{\bar{y}}^{\infty} f[x(y, \theta)] \, e^{-khy} \frac{\partial x(y, \theta)}{\partial y} \, dy. \tag{2.12}$$

As the y-plane was cut between E and M, along the portion FM we have to choose the smallest root of the equation

$$\psi(v) = y, \tag{2.13}$$

i.e., y satisfies

$$\bar{y} < y < \psi(v_0). \tag{2.14}$$

Stage 3. *Direct Inverse*

The inverse transform of $T(p)$ is obtained easily by changing the order of integration in (2.12) and at the same time changing the variable in the external integral to

$$t = hy/c. \tag{2.15}$$

Indeed

$$\begin{aligned} T(p) &= \frac{p}{c} \int_{\bar{y}}^{\infty} e^{-khy}\, dy \int_0^{\pi/2} f[x(y, \theta)] \frac{\partial x(y, \theta)}{\partial y} (-i \cos \theta)^s \cos m\theta \, d\theta \\ &= \frac{p}{h} \int_0^{\infty} e^{-pt} H\left(\frac{ht}{c} - \frac{h\bar{t}}{c}\right) dt \int_0^{\pi/2} f\left[x\left(\frac{ct}{h}, \theta\right)\right] \left[\frac{\partial x(y, \theta)}{\partial y}\right] \\ &\quad \times (-i \cos \theta)^s \cos m\theta \, d\theta, \qquad y = \frac{ct}{h}, \end{aligned} \tag{2.16}$$

where $H(u)$ is the Heaviside unit function. Thus, the inverse of $T(p)$ is

$$T(t) = \frac{1}{h} H(\tau - \bar{\tau}) \int_0^{\pi/2} f[x(\tau, \theta)] \left[\frac{\partial x(y, \theta)}{\partial y}\right]_{y=\tau} (-i \cos \theta)^s \cos m\theta \, d\theta, \tag{2.17}$$

where τ is a nondimensional time-parameter

$$\tau = ct/h \tag{2.18}$$

and

$$\bar{\tau} = y(0) = \sum_{j=0}^{N} \omega_j \lambda_j \tag{2.19}$$

Stage 4. *Change Integration Variable from θ to x*

Still in Cagniard's footsteps, we can simplify (2.17) by taking x as integration variable instead of θ, using (2.7) for $y = \tau$,

$$T(t) = \frac{1}{h} H(\tau - \bar{\tau}) \int_{x'}^{x''} f(x) \left[\frac{\partial x}{\partial y} \frac{\partial \theta}{\partial x}\right]_{y=\tau} [-i \cos \theta(x, \tau)]^s \, dx, \tag{2.20}$$

where x' and x'' correspond to $\theta = 0$ and $\theta = \pi/2$, respectively. It turns out that x' is situated either on the negative imaginary axis or in the fourth quadrant, whereas x'' is on the positive real axis so that the integration path from x' to x'' belongs entirely to the fourth quadrant. Using (2.7) we get

$$\left[\frac{\partial x}{\partial y}\,\frac{\partial \theta}{\partial x}\right]_{y=\tau} = \frac{1}{i[K(x, \tau)]^{1/2}}\,; \qquad i \cos\theta = \frac{\zeta(x, \tau)}{\rho x}, \tag{2.21}$$

where

$$K(x, \tau) = \zeta^2(x, \tau) + \rho^2 x^2; \qquad \zeta(x, \tau) = \tau - g(x). \tag{2.22}$$

The square root in (2.21) is defined so that it is real and positive when x is real and positive, uniformity being achieved by a cut in the x-plane along the segment of straight line from x' to $-x'$.

Using the expression of $\cos m\theta$ as a Chebyshev polynomial (Abramowitz and Stegun [5]) combined with (2.7) we get

$$\cos m\theta = T_m(\zeta/i\rho x) \tag{2.23}$$

and (2.20) becomes

$$T(t) = \frac{i}{h}\, H(\tau - \bar{\tau}) \int_{x''}^{x'} T_m\left(\frac{\zeta}{i\rho x}\right)\left(-\frac{\zeta}{\rho x}\right)^s \frac{f(x)\, dx}{[K(x, \tau)]^{1/2}}\,. \tag{2.24}$$

We may write (2.24) in the form

$$T(t) = \text{Complex conjugate of } \left[-\frac{i}{h} \int_{x''}^{x'} T_m\left(i\,\frac{\zeta}{\rho x}\right)\left(-\frac{\zeta}{\rho x}\right)^s \frac{f(x)\, dx}{[K(x, \tau)]^{1/2}}\right), \tag{2.25}$$

where x_1, the complex conjugate of x', is situated in the first quadrant and satisfies the equation

$$g(x) - i\rho x = \tau. \tag{2.26}$$

Expression (2.25) is obtained by using the following result:

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$$I = \int_{\Gamma} F(z)\, dz, \tag{2.27}$$

its complex conjugate is given by

$$\bar{I} = \int_{\Gamma} F(z)\, dz \tag{2.28}$$

if $F(z)$ satisfies the condition,

$$\bar{F}(z) = F(\bar{z}). \tag{2.29}$$

Now restoring the constant and taking the real or imaginary parts as shown in (2.4) we get the inverse in the form

$$\mathscr{F}_m^{1,s}(\varphi, t) = \frac{2}{\pi h} \mathscr{I}m \int_{x''}^{x'} a(m)\, i^{1-l} T_m \left(i \frac{\zeta}{\rho x}\right)\left(-\frac{\zeta}{\rho x}\right)^s \frac{x^{l+s}\varphi(x)\, dx}{[K(x, \tau)]^{1/2}}, \tag{2.30}$$

where

$$\begin{aligned} a(m) &= (-1)^{m/2} && \text{for} \quad m = \text{even} \\ &= (-1)^{(m+1)/2} && \text{for} \quad m = \text{odd}. \end{aligned} \tag{2.31}$$

Following Longman [14], we write (2.30) in a form which is more convenient for numerical calculations and also for further analytic developments as well as for physical interpretation. The main role is played by Eq. (2.26) for x_1. If

$$\tau < \tau_0 = \psi(v_0), \tag{2.32}$$

the equation for x_1 has two purely imaginary roots and we have to choose that having the least absolute value. Denoting this root by $x_1 = iv_1$, v_1 satisfies the real equation

$$\sum_{j=1}^{N} \omega_j(\lambda_j^2 - v_1^2)^{1/2} + \rho v_1 = \tau. \tag{2.33}$$

Taking into account the assumptions on $\psi(x)$, we find that for $\bar{y} < \tau < \tau_0$ we get from (2.30)

$$\mathscr{F}_m^{1,s}(\varphi, t) = -\frac{2}{\pi h} \int_{\bar{\lambda}}^{v_1} T_m \left(\frac{\zeta}{\rho v}\right) \mathscr{I}m \left[a(m)\, i^{-l} \left(-\frac{\zeta}{i\rho v}\right)^s (iv)^{l+s} \varphi(iv)\right] \frac{dv}{[K(iv\tau)]^{1/2}},$$

where

$$\bar{\lambda} = \min(\lambda, \bar{\lambda}_1, ..., \bar{\lambda}_k). \tag{2.35}$$

It may happen however that $v_1 < \bar{\lambda}$. In this case there is no contribution for $\tau < \tau_0$. This is certainly the case if $v_0 < \bar{\lambda}$, i.e., when $\rho < \rho_1$, ρ_1 being defined as follows:

$$\rho_1 = \sum_{j=1}^{N} \frac{\omega_j \bar{\lambda}}{(\lambda_j^2 - \bar{\lambda}^2)^{1/2}}. \tag{2.36}$$

In such a case $\mathscr{F}_m^{1,s}(\varphi, t) = 0$ for $\tau < \tau_0$. For $\tau > \tau_0$, $\mathscr{F}_m^{1,s}(\varphi, t)$ is given by (2.30) where x'' can be replaced by $i\bar{\lambda}$ since there is no contribution to the integral from the portion of the real axis between the origin and x'' and from the portion of the imaginary axis between the origin and $i\bar{\lambda}$, as the integrand is real.

When $\rho > \rho_1$ there are two cases: either $v_1 < \bar{\lambda}$ and in this case the result is zero for $\tau < \tau_0$. This condition may be translated into a condition on τ, by using the function $\psi(v)$: $\tau < \tau^*$, where

$$\tau^* = \psi(\bar{\lambda}) = g(\bar{\lambda}) + \rho\bar{\lambda}. \tag{2.37}$$

Thus, when $\tau^* < \tau < \tau_0$, $\mathscr{F}_m^{1,s}(\varphi, t)$ is given by (2.34) whereas when $\tau > \tau_0$, $\mathscr{F}_m^{1,s}(\varphi, t)$ is again given by (2.30).

Summarizing, the inverse of (1.1) for $n = 1$, $s = 0$ is obtained as follows:

For $\bar{\lambda} \geqslant \lambda$ or $\bar{\lambda} < \lambda$ and $\rho < \rho_1$,

$$\begin{aligned} \mathscr{F}_m^{1,s}(\varphi, t) &= 0 && \tau < \tau_0 \\ &= Y_m^{1,s}(\tau) && \tau > \tau_0. \end{aligned} \tag{2.38}$$

For $\bar{\lambda} < \lambda$ and $\rho > \rho_1$,

$$\begin{aligned} \mathscr{F}_m^{1,s}(\varphi, t) &= 0 && \tau < \tau^* \\ &= X_m^{1,s}(\tau) && \tau^* < \tau < \tau_0 \\ &= Y_m^{1,s}(\tau) && \tau > \tau_0, \end{aligned} \tag{2.39}$$

where $X_m^{1,s}(\tau)$ and $Y_m^{1,s}(\tau)$ are the right-hand sides of (2.34) and (2.30), respectively,

$$X_m^{1,s}(\tau) = -\int_{\bar{\lambda}}^{v_1} v^l \, \mathscr{Im}[\alpha_m(iv)] \, \beta_m^{1,s}(iv, \tau) \, dv, \tag{2.40}$$

$$Y_m^{1,s}(\tau) = \mathscr{Im} \int_{i\bar{\lambda}}^{x_1} i^{1-l} x^l \alpha_m(x) \, \beta_m^{1,s}(x, \tau) \, dx, \tag{2.41}$$

where

$$\alpha_m(x) = \frac{2}{\pi h} \, a(m) \, \varphi(x) \tag{2.42}$$

$$\beta_m^{1,s}(x, y) = T_m\left(\frac{i\zeta}{\rho x}\right)\left[-\frac{\zeta(x, y)}{\rho}\right] [K(x, y)]^{-1/2}. \tag{2.43}$$

There is a direct physical interpretation for relations (2.38)–(2.39), connected with the fact that the constants λ_j and $\bar{\lambda}_k$ are ratios between a characteristic wave-velocity and various wave-velocities in the structure. The quantity ρ_1 is the least horizontal distance at which total reflection can take place. If the horizontal distance is less than ρ_1 no totally reflected wave is possible so that only a reflected wave arrives, the arrival time being τ_0. However, if the horizontal distance is larger than ρ_1 a wave arrives before τ_0: This wave may be interpreted as being reflected at the critical angle, traveling on a separating interface at a layer velocity corresponding to the lower medium and then returning to the receiver directly or after a number of reflection. The geometric minimum arrival time of such a ray is τ^*. Such a wave is possible only if in the lower medium there is a wave-velocity higher than the velocities in the higher medium and this is the meaning of the condition $\bar{\lambda} < \lambda$.

3. The Inverse Transform for $n = 0$

As

$$\mathscr{F}_m^{0,s}(\varphi, p) = (1/k)\, \mathscr{F}_m^{1,s}(\varphi, p) \tag{3.1}$$

it follows that the inverse transform of $\mathscr{F}_m^{0,s}$ is obtained from $\mathscr{F}_m^{1,s}$ by integrating with respect to t,

$$\mathscr{F}_m^{0,s}(\varphi, t) = h \int_0^{\tau} \mathscr{F}_m^{1,s}(\varphi, y)\, dy. \tag{3.2}$$

We need therefore the following integrals:

$$X_m^{0,s}(\tau) = h \int_{\tau^*}^{\tau} X_m^{1,s}(y)\, dy, \qquad Y_m^{0,s}(\tau) = h \int_{\tau_0}^{\tau} Y_m^{1,s}(y)\, dy, \tag{3.3}$$

and

$$X_m^{0,s}(\tau_0) + Y_m^{0,s}(\tau) = h \int_{\tau^*}^{\tau_0} X_m^{1,s}(y)\, dy + h \int_{\tau_0}^{\tau} Y_m^{1,s}(y)\, dy. \tag{3.4}$$

First calculate $Y_m^{0,s}(\tau)$:

$$\begin{aligned} Y_m^{0,s}(\tau) = A + B = -h \int_{\tau_0}^{\tau} dy\, \mathscr{I}m \int_{\bar{\lambda}}^{v_0} v^l \alpha_m(iv)\, \beta_m^{n,s}(iv, y)\, dv \\ + h \int_{\tau_0}^{\tau} dy\, \mathscr{I}m \int_{iv_0}^{x_1(y)} i^{1-l} x^l \alpha_m(x)\, \beta_m^{r,s}(x, y)\, dx. \end{aligned} \tag{3.5}$$

If $\rho < \rho_1$, the integrand of A is real so that there is no contribution from this term. When $\rho > \rho_1$, the first term may be calculated by interchanging the order of integration

$$A = -\, \mathscr{I}m\, h \int_{\bar{\lambda}}^{v_0} v^l \alpha_m(iv)\, dv \int_{\tau_0}^{\tau} \beta_m^{1,s}(iv, y)\, dy. \tag{3.6}$$

Denote by $\tilde{\beta}_m^{1,s}$ an indefinite integral of $\beta_m^{1,s}(x, y)$ with respect to y,

$$\frac{d\tilde{\beta}_m^{1,s}}{dy}(x, y) = \beta_m^{1,s}(x, y). \tag{3.7}$$

Then

$$A = -\mathscr{I}m\, h \int_{\bar{\lambda}}^{v_0} v^l \alpha_m(iv)[\tilde{\beta}_m^{1,s}(iv,\tau) - \tilde{\beta}_m^{1,s}(iv,\tau_0)\, dv. \tag{3.8}$$

Now calculate B. When y varies from τ_0 to τ, the point $x_1(y)$ describes a curve in the first quadrant of the x-plane,

$$x = x(\sigma). \tag{3.9}$$

Here x is the solution of Eq. (2.26) with τ replaced by σ in the right-hand side,

$$g(x) - i\rho x = \sigma, \tag{3.10}$$

σ varying between τ_0 and y. Using σ as an integration variable, which is possible as (3.10) is one-to-one, we get

$$B = h \int_{\tau_0}^{\tau} dy\, \mathscr{I}m \int_{\tau_0}^{y} i^{1-l} x^l(\sigma)\, \alpha_m[x(\sigma)]\, \beta_m^{1,s}[x(\sigma), y]\, \frac{\partial x}{\partial \sigma}\, d\sigma \tag{3.11}$$

and interchanging the order of integration,

$$\begin{aligned} B &= \mathscr{I}m\, h \int_{\tau_0}^{\tau} i^{1-l} x^l(\sigma)\, \alpha_m[x(\sigma)]\, \frac{\partial x}{\partial \sigma}\, d\sigma \int_{\sigma}^{\tau} \beta_m^{1,s}[x(\sigma), y]\, dy \\ &= \mathscr{I}m\, h \int_{\tau_0}^{\tau} i^{1-l} x^l(\sigma)\, \alpha_m[x(\sigma)]\{\tilde{\beta}_m^{1,s}[x(\sigma),\tau] - \tilde{\beta}_m^{1,s}[x(\sigma),\sigma]\}\, \frac{\partial x}{\partial \sigma}\, d\sigma. \end{aligned} \tag{3.12}$$

Going back to x as integration variable, we get

$$B = \mathscr{I}m\, h \int_{iv_0}^{x_1(\tau)} i^{1-l} x^l \alpha_m(x)\{\tilde{\beta}^{1,s}(x,\tau) - \tilde{\beta}_m^{1,s}[x, g(x) - i\rho x]\}\, dx. \tag{3.13}$$

Returning to (3.5) and using (3.8) and (3.13)

$$\begin{aligned} Y_m^{0,s}(\tau) &= \mathscr{I}m\, h \int_{\bar{\lambda}_i}^{x_1(\tau)} i^{1-l} x^l \alpha_m(x)\, \tilde{\beta}_m^{1,s}(x,\tau)\, dx + \mathscr{I}m \int_{\bar{\lambda}}^{v_0} v^l \alpha(iv)\, \tilde{\beta}_m^{1,s}(iv,\tau_0) \\ &\quad - \mathscr{I}m\, h \int_{iv_0}^{x_1(\tau)} i^{1-l} x^l \alpha_m(x)\, \tilde{\beta}_m^{1,s}[x, g(x) - i\rho x]\, dx. \end{aligned} \tag{3.14}$$

Now calculate $X_m^{0,s}(\tau)$,

$$X_m^{0,s}(\tau) = -h \int_{\tau^*}^{\tau} dy \int_{\bar{\lambda}}^{v_1(y)} v^l\, \mathscr{I}m[\alpha_m(iv)]\, \beta_m^{1,s}(iv, y)\, dv. \tag{3.15}$$

As v_1, being the smallest root of $\psi'(v)$, is less than v_0 and increases with τ for $\tau < \tau_0$, we can reverse the order of integration in (3.15), using (2.33) with τ replaced by y,

$$X_m^{0,s}(\tau) = -h \int_{\bar{\lambda}}^{v_1(\tau)} v^l \, \mathscr{I}m[\alpha_m(iv)] \, dv \int_{g(iv_1)+\rho v_1}^{\tau} \beta_m^{1,s}(iv, y) \, dy, \tag{3.16}$$

i.e.,

$$X_m^{0,s}(\tau) = -h \int_{\bar{\lambda}}^{v_1(\tau)} v^l \, \mathscr{I}m[\alpha_m(iv)]\{\tilde{\beta}_m^{1,s}(iv, \tau) - \tilde{\beta}_m^{1,s}[iv, g(iv_1) + \rho v_1]\} \, dv. \tag{3.17}$$

Finally, calculate (3.4) using (3.14) and (3.17)

$$X_m^{0,s}(\tau_0) + Y_m^{0,s}(\tau) = \mathscr{I}m \, h \int_{\bar{\lambda} i}^{x_1(\tau)} i^{1-l} x^l \alpha_m(x)\{\tilde{\beta}_m^{1,s}(x, \tau) - \tilde{\beta}_m^{1,s}[x, g(x) - i\rho x]\} \, dx. \tag{3.18}$$

Thus, the inverse transform $\mathscr{F}_m^{0,s}(\varphi, t)$ is given by formulas similar to (2.38)–(2.39), the functions $X_m^{1,s}(\tau)$, $Y_m^{1,s}(\tau)$ being replaced by

$$X_m^{0,s}(\tau) = -h \int_{\bar{\lambda}}^{v_1} v^l \, \mathscr{I}m[\alpha_m(iv)] \, \beta_m^{0,s}(iv, \tau) \, dv, \tag{3.19}$$

$$Y_m^{0,s}(\tau) = \mathscr{I}m \, h \int_{\bar{\lambda} i}^{x} i^{1-l} x^l \alpha_m(x) \, \beta_m^{0,s}(x, \tau) \, dx, \tag{3.20}$$

where

$$\beta_m^{0,s}(x, y) = \tilde{\beta}_m^{1,s}(x, y) - \tilde{\beta}_m^{1,s}[x, g(x) - i\rho x]. \tag{3.21}$$

4. The Inverse Transform for $n < 0$

The procedure for obtaining the inverse transform for $n = 0$ can be repeated several times. Therefore, the inverse for any $n < 0$ is obtained inductively in the form (2.38)–(2.39) with $X_m^{1,s}(\tau)$, $Y_m^{1,s}(\tau)$ replaced by

$$X_m^{n,s}(\tau) = -h^{-n+1} \int_{\bar{\lambda}}^{v_1} v^l \, \mathscr{I}m[\alpha_m(iv)] \, \beta_m^{n,s}(iv, \tau) \, dv, \tag{4.1}$$

$$Y_m^{n,s}(\tau) = \mathscr{I}m \, h^{-n+1} \int_{\bar{\lambda} i}^{x_1} i^{1-l} x^l \alpha_m(x) \, \beta_m^{n,s}(x, \tau) \, dx, \tag{4.2}$$

where

$$\beta_m^{n,s}(x, y) = \tilde{\beta}_m^{n+1,s}(x, y) - \tilde{\beta}_m^{n+1,s}[x, g(x) - i\rho x]. \tag{4.3}$$

Thus, in order to write the inverse transform for a negative value of n, all one needs is to calculate several indefinite integrals, the inverse being expressed again by single integrals. In what follows we consider several illustrative examples [4].

EXAMPLE 4.1. *Strike-slip point source with linear time-dependence.* (a) The vertical displacement is a sum of generalized rays of the form $\mathscr{F}_2^{1,0}(\varphi, p)$, i.e., corresponding to $m = 2, n = 1, s = 0$, so that

$$T_2(\cos\theta) = 2\cos^2\theta - 1 \tag{4.4}$$

and therefore

$$T_2\left(\frac{i\zeta}{\rho x}\right) = -\frac{2[\tau - g(x)]^2 + \rho^2 x^2}{\rho^2 x^2} \tag{4.5}$$

leading to

$$X_2^{1,0}(\tau) = \frac{2}{\pi h \rho^2} \int_{\bar\lambda}^{v_1} \mathscr{I}m[\varphi(iv)] \frac{2[\tau - g(iv)]^2 - \rho^2 v^2}{v[K(iv, \tau)]^{1/2}}\, dv, \tag{4.6}$$

$$Y_2^{1,0}(\tau) = \frac{2}{\pi h \rho^2} \mathscr{I}m \int_{i\bar\lambda}^{x_1} \varphi(x) \frac{2[\tau - g(x)]^2 + \rho^2 x^2}{x[K(x, \tau)]^{1/2}}\, dx. \tag{4.7}$$

(b) The radial displacement containes generalized rays of the form $\mathscr{F}_2^{1,1}(\varphi, p)$. Thus, in this case $m = 2$, $n = 1$, $s = 1$. $X_2^{1,1}(\tau)$ and $Y_2^{1,1}(\tau)$ coincide with $X_2^{1,0}$ and $Y_2^{1,0}$, respectively, except for the factor $-[\tau - g(x, \tau)]/\rho$.

(c) The azimuthal displacement component contains generalized rays of the form $\mathscr{F}_2^{0,0}(\varphi, p)$, i.e., $m = 2$, $n = 0$, $s = 0$. As

$$\beta_2^{1,0}(x, y) = -\frac{2[y - g(x)]^2 + \rho^2 x^2}{\rho^2 x^2([y - g(x)]^2 + p^2 x^2)^{1/2}} \tag{4.8}$$

we have, according to (2.43) and (4.3),

$$\bar\beta_2^{1,0}(x, y) = -\frac{y - g(x)}{\rho^2 x^2}([y - g(x)]^2 + \rho^2 x^2)^{1/2} = \beta_2^{0,0}(x, y) \tag{4.9}$$

and therefore

$$X_2^{0,0}(\tau) = \frac{2}{\pi\rho^2} \int_{\bar\lambda}^{v_1} \frac{\tau - g(iv)}{v} \mathscr{I}m[\varphi(iv)][K(iv, \tau)\, dv]^{1/2}, \tag{4.10}$$

$$Y_2^{0,0}(\tau) = \frac{2}{\pi\rho^2} \mathscr{I}m \int_{i\bar\lambda}^{x_1} \frac{\tau - g(x)}{x} \varphi(x)[K(x, \tau)\, dx]^{1/2}. \tag{4.11}$$

EXAMPLE 4.2. *Dip-slip point-source* with *linear time-dependence*. The displacement components contain generalized rays of the form $\mathscr{F}_m^{n,s}$ for three different cases:

(a) $m = 1, n = 1, s = 0$.

We have

$$T_1\left(\frac{i\zeta}{\rho x}\right) = \frac{i\zeta}{\rho x} \tag{4.12}$$

and

$$\alpha_1(x) = -\frac{2}{\pi h}\varphi(x), \qquad \beta_1^{1,0}(x, y) = \frac{i\zeta}{\rho x}[K(x, y)]^{-1/2} \tag{4.13}$$

leading to

$$X_1^{1,0}(\tau) = \frac{2}{\pi\rho h}\int_{\bar\lambda}^{v_1} \mathscr{I}m[\varphi(iv)]\frac{\tau - g(iv)}{v[K(iv, \tau)]^{1/2}}\,dv, \tag{4.14}$$

$$Y_1^{1,0}(\tau) = \frac{2}{\pi\rho h}\mathscr{I}m\int_{i\bar\lambda}^{x_1}\varphi(x)\frac{\tau - g(x)}{x[K(x, \tau)]^{1/2}}\,dx. \tag{4.15}$$

(b) $m = 1, n = 1, s = 1$.

$X_1^{1,1}$ and $Y_1^{1,1}$ are identical with (4.14)–(4.15), respectively, except for the factor $[\tau - g(x, \tau)]/\rho$.

(c) $n = 1, n = 0, s = 0$.

From (4.13) we get immediately

$$\tilde\beta_1^{1,0}(x, y) = \frac{i}{\rho x}[K(x, y)]^{1/2} = \beta_1^{0,0}(x, y) \tag{4.16}$$

so that

$$X_1^{0,0}(\tau) = \frac{2}{\pi\rho}\int_{\bar\lambda}^{v_1}\mathscr{I}m[\varphi(iv)][K(iv, \tau)]^{1/2}\frac{dv}{v}, \tag{4.17}$$

$$Y_1^{0,0}(\tau) = \frac{2}{\pi\rho}\mathscr{I}m\int_{i\bar\lambda}^{x_1}\varphi(x)[K(x, \tau)]^{1/2}\frac{dx}{x}. \tag{4.18}$$

EXAMPLE 4.3. *Strike-slip point-source with quadratic time-dependence.* (a) The vertical displacement corresponds now to the values $m = 2$, $n = 0$, $s = 0$, i.e., are given by (4.10)–(4.11).

(b) The radial displacement corresponds to $m = 2$, $n = 0$, $s = 1$. We have

$$\beta_2^{1,1}(x, y) = \frac{\{2[y - g(x)]^2 + \rho^2x^2\}[y - g(x, y)]}{\rho^3x^2([y - g(x)]^2 + \rho^2x^2)^{1/2}} \tag{4.19}$$

so that

$$\tilde\beta_2^{1,1}(x, y) = \frac{1}{3\rho^2x^2}\{2[y - g(x)]^2 - \rho^2x^2\}([y - g(x)]^2 + \rho^2x^2)^{1/2} = \beta_2^{0,1}(x, y), \tag{4.20}$$

i.e.

$$X_2^{0,1}(\tau) = -\frac{2}{3\pi\rho^3}\int_{\bar{\lambda}}^{v_1} \mathscr{I}m[\varphi(iv)]\{2[\tau - g(iv)] + \rho^2v^2\}([\tau - g(iv)]^2 - \rho^2v^2)^{1/2}\frac{dv}{v}, \tag{4.21}$$

$$Y_2^{0,1}(\tau) = \frac{2}{3\pi\rho^2}\mathscr{I}m\int_{\bar{\lambda}i}^{x_1} \varphi(x)\{2[\tau - g(x)] - \rho^2x^2\}([\tau - g(x)]^2 + \rho^2x^2)^{1/2}\frac{dx}{x}. \tag{4.22}$$

(c) The azimuthal displacement corresponds to $m = 2$, $n = -1$, $s = 0$. Using (4.9) we get

$$\bar{\beta}_2^{0,0}(x, y) = -\frac{1}{3\rho^2x^2}\{[y - g(x)]^2 + \rho^2x^2\}^{3/2} = \beta_2^{-1,0} \tag{4.23}$$

so that

$$X_2^{-1,0}(\tau) = -\frac{2h}{3\pi\rho^2}\int_{\bar{\lambda}}^{v_1}\frac{\mathscr{I}m\,\varphi(iv)}{v}\{[\tau - g(iv)]^2 - \rho^2v^2\}^{3/2}\,dv, \tag{4.24}$$

$$Y_2^{-1,0}(\tau) = \frac{2h}{3\pi\rho^2}\mathscr{I}m\int_{\bar{\lambda}i}^{x_1}\frac{\varphi(x)}{x}\{[\tau - g(x)]^2 + \rho^2x^2\}^{3/2}\,dx. \tag{4.25}$$

5. The Inverse Transform for $n > 1$

In many applications the inverse transform of expressions involving $\mathscr{F}_m^{n,s}(\varphi, p)$ is required for $n = 2$, e.g., in cases when the displacement for Heaviside point-sources is needed. When velocities or accelerations are needed for linear or Heaviside sources, we have to calculate $\mathscr{F}_m^{n,s}(\varphi, p)$ with $n > 2$.

For $n = 2$

$$\mathscr{F}_m^{2,s}(\varphi, p) = k\mathscr{F}_m^{1,s}(\varphi, p) \tag{5.1}$$

so that the inverse Transform of $\mathscr{F}_m^{2,s}(\varphi, p)$ is obtained by differentiating with respect to τ,

$$\mathscr{F}_m^{2,s}(\varphi, t) = \frac{1}{h}\frac{d}{d\tau}\mathscr{F}_m^{1,s}(\varphi, t), \tag{5.2}$$

i.e., $\mathscr{F}_m^{2,s}(\varphi, t)$ is given by (2.38)–(2.39) with $X_m^{1,s}$ and $Y_m^{1,s}(\tau)$ replaced by their derivatives with respect to τ divided by h. These derivatives can be obtained analytically from (2.40)–(2.41) through the following changes of variables, used by Longman [14] to remove the singularity in the integrands for numerical purposes.

$$v = v_1(1 - u^2) \tag{5.3}$$

for (2.40) and

$$x = x_1(1 - u^2) \tag{5.4}$$

for (2.41). Making these changes of variables after extending the integration to the origin, we get integrals with fixed end-points, so that we can differentiate under the integral sign.

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References

1. F. Abramovici, *Bull. Seismol. Soc. Amer.* **60** (1970), 1861.
2. F. Abramovici and Z. Alterman, in "Methods of Computational Physics," Vol. 4, p. 349, Academic Press, New York/London, 1965.
3. F. Abramovici and J. Gal-Ezer, *Bull. Seismol. Soc. Amer.* **68** (1978), 81.
4. F. Abramovici and J. Gal-Ezer, in preparation.
5. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Tables," Dover, New York, 1965.
6. A. Ben-Menahem and M. Vered, *Bull. Seismol. Soc. Amer.* **63** (1973), 971.
7. L. Cagniard, "Réflexion et Réfraction des Ondes Seismiques Progressives," Gauthier–Villars, Paris, 1939.
8. U. Chandra, *Bull. Seismol. Soc. Amer.* **59** (1969), 317.
9. D. V. Helmberger, *Bull. Seismol. Soc. Amer.* **58** (1968), 179.
10. A. T. de Hoop, "Representation Theorems for the Displacement in an Elastic Solid and Their Application to Elastodynamic Diffraction Theory," Doctoral dissertation, Delft, 1958.
11. A. T. de Hoop, *Appl. Sci. Res.* **8** (1960), 349.
12. M. Israel and M. Vered, *Bull. Seismol. Soc. Amer.* **67** (1977), 631.
13. M. Israel and R. Kovach, *Bull. Seismol. Soc. Amer.* **67** (1977), 977.
14. I. M. Longman, *J. Acoust. Soc. Amer.* **33** (1961), 954.
15. G. Müller, *Z. Geophys.* **34** (1968), 15.
16. G. Müller, *Z. Geophys.* **34** (1968), 147.
17. G. Müller, *Z. Geophys.* **35** (1969), 347.
18. C. L. Pekeris, *Proc. Nat. Acad. Sci. U.S.* **26** (1940), 433.
19. C. L. Pekeris, *Proc. Nat. Acad. Sci. U.S.* **41** (1955), 469.
20. C. L. Pekeris, *Proc. Nat. Acad. Sci. U.S.* **41** (1955), 629.
21. C. L. Pekeris, *Proc. Nat. Acad. Sci. U.S.* **42** (1956), 439.
22. C. L. Pekeris, Z. Alterman, and F. Abramovici, *Bull. Seismol. Soc. Amer.* **53** (1963), 39.
23. C. L. Pekeris, Z. Alterman, F. Abramovici, and H. Jarosh, *Rev. Geophys.* **3** (1965), 25.
24. J. W. C. Sherwood, *Proc. Phys. Soc. London* **71** (1958), 207.
25. M. Vered and A. Ben-Menahem, *Bull. Seismol. Soc. Amer.* **64** (1974), 1221.